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Rational three-spin string duals and non-anomalous finite size effects

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Abstract: We determine by a one line computation the one-loop conformal dimension and the associated non-anomalous finite size correction for all operators dual to spinning strings of rational type having three angular momenta (J_1, J_2, J_3) on S^5. Finite size corrections are conjectured to encode information about string sigma model loop corrections to the spectrum of type-IIB superstrings on $AdS_5 \times S^5$. We compare our result to the zero-mode contribution to the leading quantum string correction derived for the stable three-spin string with two out of the three spin labels identical and observe agreement. As a side result we clarify the relation between the Bethe root description of three-spin strings of the type (J, J', J') with respectively $J > J'$ and $J < J'$.

Keywords: Duality in Gauge Field Theories, AdS-CFT and dS-CFT Correspondence, Bethe Ansatz.

Contents

1. Introduction

Semi-classical analysis of strings propagating on $AdS_5 \times S^5$, see [1] for a review, has provided us with very concrete realizations of the AdS/CFT duality [2]. The prime example is the matching of the energies of free strings with conformal dimensions of operators of planar $\mathcal{N} = 4$ SYM. Strings amenable to semi-classical analysis are strings carrying large quantum numbers. For strings with several large angular momenta on $\mathrm{AdS}_5 \times S^5$, at least one of them lying on S^5, the semi-classical analysis is particularly clean [3, 4]. The classical string energy organizes into a power series in λ/J^2 where J is the sum of angular momenta and λ is the squared string tension which via the AdS/CFT dictionary is mapped onto the 't Hooft coupling of the dual gauge theory. Furthermore, string sigma model loop corrections are suppressed by powers of $1/J$ compared to the classical energy. More precisely, the combined expansion of the semi-classical energy in λ/J^2 and J takes the form [3, 4]

$$E = J\left\{1 + \frac{\lambda}{J^2}\left(E_1^{(0)} + \frac{1}{J}E_1^{(1)} + \cdots\right) + \left(\frac{\lambda}{J^2}\right)^2\left(E_2^{(0)} + \frac{1}{J}E_2^{(1)} + \cdots\right) + \cdots\right\}, \quad (1.1)$$

where the classical energy is given by

$$E_{cl} = J\left\{1 + \frac{\lambda}{J^2}E_1^{(0)} + \left(\frac{\lambda}{J^2}\right)^2 E_2^{(0)} + \cdots\right\}. \quad (1.2)$$

Based on the AdS/CFT conjecture one expects a similar reorganization of the perturbative series for the conformal dimension of the dual operator to be possible. In gauge theory language angular momenta are simply representation labels and J is the sum of such labels. It is thus tempting to attempt a comparison with string theory, calculating conformal dimensions by first doing a perturbative expansion and subsequently taking the limit $J \to \infty$ with λ/J^2 fixed. This approach was very successful at the one-loop level where the dilatation operator of the gauge theory could be proved identical to the hamiltonian of an integrable spin chain [5 − 7] and diagonalization could be carried out in a number of specific cases using Bethe equation techniques [8 − 12]. It also became possible to prove the equivalence of the semi-classical treatment of the string and the perturbative treatment of the gauge theory at order λ at a more general level, not referring to particular solutions [13 − 21]. Encouraged by this success there are several paths one can take.

One can try to incorporate higher order terms in the perturbative analysis of the gauge theory. This should allow one to reproduce more terms in the expansion (1.2) of the classical string energy. As the dilatation operator of the gauge theory is known to higher loop orders in certain sub-sectors [22, 23], see also [24], this line of investigation is indeed possible and was initiated in [25]. The matching with the semi-classical string analysis worked successfully at two loops but at three loops a discrepancy was observed [25], see also [26]. An explanation of this discrepancy as nothing but a manifestation of the strong/weak coupling nature of the AdS/CFT correspondence has been put forward [25, 27].

Another path one can take to further investigate the relation between $\mathcal{N} = 4$ SYM and semi-classical strings is to attempt to go beyond the planar approximation of the gauge theory. In string theory language this implies taking into account string interactions. While an interesting suggestion for how to deal semi-classically with the splitting of certain strings propagating on $AdS_5 \times S^5$ exists [28], unfortunately not much progress has been made in the development of the necessary calculational techniques on the gauge theory side.

Finally, one could attempt to study the gauge theory counterpart of string sigma model loop corrections and this is the line of investigation we shall follow here. The one-loop string correction has been known for some time for one particular string configuration, namely a circular string rotating on S^5 and carrying centre of mass angular momentum J_1 and two equal angular momenta $J_2 = J_3$ with respect to two orthogonal planes in S^5. The string is stable for J_2 small enough and in the region of stability the one loop correction has been calculated [3, 4, 29]. Actually, the calculation of this one-loop correction provided the rationale for all subsequent semi-classical analysis of strings propagating in $AdS_5 \times S^5$. On the gauge theory side, assuming the dilatation operator to be given by the hamiltonian of an integrable spin chain, quantum corrections to the classical string energy translates into finite size corrections to energy eigenvalues of the infinite chain. In the present paper we shall determine the non-anomalous part of the finite size correction to the conformal dimension (i.e. spin chain eigen energy) for the operator dual to the above mentioned three-spin string. Actually, we shall do much more than that. We shall determine the non-anomalous part of the finite size correction for all operators dual to so-called rational strings carrying three angular momenta (J_1, J_2, J_3) on S^5. We shall furthermore compare the result obtained for the dual of the stable three-spin string above to the zero-mode contribution to the string quantum correction and find agreement. So far non-anomalous finite size contributions were only determined for the simpler case of the dual of an unstable string with two spins on S^5 [30] and the case of a stable string with one spin on S^5 and one on AdS_5 [19]. Very recently the one-loop string correction for the latter string was determined and agreement between zero-mode contribution and non-anomalous finite size effects in gauge theory likewise found [31].

In section 2 we shall review the properties of strings rotating on S^5 and their gauge theory duals. After that in section 3 we calculate the non-anomalous part of the finite size correction to the conformal dimensions of operators dual to rational three-spin strings and compare in section 4 to the known result for the stable three-spin string with two out of the three angular momenta coinciding. Subsequently, in section 5 we explain how one can easily obtain also all higher charges of the integrable spin chain in question and the associated non-anomalous finite size corrections. Finally in section 6 we discuss some particular points of our solution. Section 7 contains our conclusions.

After the completion of this manuscript we received [32] where it was demonstrated that for a two-spin string with one large angular momentum on S^5 and one on AdS_5 the finite size corrections have a non-anomalous part which matches the contribution to the string sigma model loop correction coming from non-zero modes.

2. Strings spinning on S^5 and their gauge theory duals

One class of strings spinning on S^5 is particularly simple, namely the class of circular, rigid strings carrying centre of mass angular momentum J_1 and two equal angular momenta $J_2 = J_3$ with respect to two orthogonal planes in S^5. Denoting by $J = J_1 + J_2 + J_3$ the total angular momentum and by k the winding number the first terms in the expansion (1.2) of the classical energy of such strings take the form [3, 4]

$$E_{cl} = J + \lambda\, k^2 \frac{J_2}{J^2}\,, \tag{2.1}$$

The above strings form a sub-class of the so-called rational three-spin strings [33]. A general three-spin string of rational type with angular momenta (J_1, J_2, J_3) is characterized by the angular momenta being related by

$$m_1 J_1 + m_2 J_2 + m_3 J_3 = 0\,, \tag{2.2}$$

with m_1, m_2 and m_3 integer and by the first terms in the large-J expansion of the classical energy being given by

$$E_{cl} = J + \frac{\lambda}{2J} \sum_{i=1}^{3} m_i^2 \frac{J_i}{J}\,. \tag{2.3}$$

The simpler case with $J_2 = J_3$ is recovered for $m_1 = 0$, $m_2 = -m_3 = k$. In addition to the rational type, three-spin strings come in an elliptic and in a hyper-elliptic version, the terminology referring to the type of functions needed to parametrize the classical string sigma model solution [34]. Only for the simplest rational three-spin string with angular momenta $(J_1, J_2, J_3) = (J_1, J_2, J_2)$ a study of one-loop sigma model corrections has been possible [3, 4, 29].

At one loop order in λ rigid strings spinning on S^5 with angular momenta (J_1, J_2, J_3) are dual to operators of the type $\mathrm{Tr}(X^{J_1}Y^{J_2}Z^{J_3} + perm's)$, where X, Y and Z are the three complex scalars of $\mathcal{N} = 4$ SYM, and the conformal dimension of such operators can be found by diagonalizing the hamiltonian of the ferromagnetic SU(3) $\subset$ SO(6) spin chain of length $J = J_1 + J_2 + J_3$. The diagonalization is carried out by solving a set of algebraic equations for the Bethe roots [35]. Representations of SO(6) can be labelled by three highest weight labels (J_1, J_2, J_3) with $J_1 \geq J_2 \geq J_3$ or equivalently by three Dynkin indices $[d_1, d_2, d_3] = [J_2 - J_3, J_1 - J_2, J_2 + J_3]$. For the SU(3) spin chain there are two types of Bethe roots $\{u_{1,j}\}_{j=1}^{n_1}$ and $\{u_{2,j}\}_{j=1}^{n_2}$ and the number of these, n_1 and n_2, determine the representation as follows

$$[d_1, d_2, d_3] = [n_1 - 2n_2, J - 2n_1 + n_2, n_1]\,, \tag{2.4}$$

which implies[1]

$$(J_1, J_2, J_3) = (J - n_1, n_1 - n_2, n_2)\,. \tag{2.5}$$

The Bethe equations read

$$\left(\frac{u_{1,j} + i/2}{u_{1,j} - i/2}\right)^J = \prod_{k\neq j}^{n_1} \frac{u_{1,j} - u_{1,k} + i}{u_{1,j} - u_{1,k} - i} \prod_{k=1}^{n_2} \frac{u_{1,j} - u_{2,k} - i/2}{u_{1,j} - u_{2,k} + i/2}, \tag{2.6}$$

$$1 = \prod_{k\neq j}^{n_2} \frac{u_{2,j} - u_{2,k} + i}{u_{2,j} - u_{2,k} - i} \prod_{k=1}^{n_1} \frac{u_{2,j} - u_{1,k} - i/2}{u_{2,j} - u_{1,k} + i/2}\,. \tag{2.7}$$

The spin chain eigenstates sought for must reflect the characteristics of a trace implying first that the spin chain must be considered periodic and secondly that a zero momentum condition must be imposed, i.e.

$$1 = \prod_{j=1}^{n_1} \left(\frac{u_{1,j} + i/2}{u_{1,j} - i/2}\right). \tag{2.8}$$

A solution of the Bethe equations supplemented by (2.8) gives rise to an eigen operator with one-loop conformal dimension

$$\gamma_1 = \frac{J}{8\pi^2} \sum_{j=1}^{n_1} \frac{1}{(u_{1,j})^2 + 1/4}\,, \tag{2.9}$$

where in analogy with eq. (1.1) we have for the conformal dimension

$$\gamma = J\left\{1 + \frac{\lambda}{J^2}\gamma_1 + \left(\frac{\lambda}{J^2}\right)^2 \gamma_2 + \ldots\right\}. \tag{2.10}$$

[1] We use the notation (J_1, J_2, J_3) both for representation labels on the gauge theory side and for angular momenta on the string theory side. When referring to gauge theory quantities it is always assumed that $J_1 \geq J_2 \geq J_3$ whereas in general no ordering is assumed when referring to string theory quantities.

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Rescaling all roots by a factor of J, i.e. setting $u_{i,j} = J\, q_{i,j}$, taking the logarithm of the equations (2.6) and (2.7), and expanding for large J one finds

$$\frac{1}{q_{1,k}} + 2\pi n_k = \frac{2}{J}\sum_{l\neq k}^{n_1}\frac{1}{q_{1,k}-q_{1,l}} - \frac{1}{J}\sum_{l=1}^{n_2}\frac{1}{q_{1,k}-q_{2,l}}\,, \tag{2.11}$$

$$2\pi m_k = \frac{2}{J}\sum_{l\neq k}^{n_2}\frac{1}{q_{2,k}-q_{2,l}} - \frac{1}{J}\sum_{l=1}^{n_1}\frac{1}{q_{2,k}-q_{1,l}}\,. \tag{2.12}$$

Solving these equations to leading and next to leading order in $1/J$ gives the one-loop conformal dimension and the associated non-anomalous part of its $1/J$ correction. An additional anomalous piece originating from nearby Bethe roots with $q_j - q_k \sim \mathcal{O}(\frac{1}{J})$ is not taken into account by these formulas [32]. By the same procedure we get for the zero momentum condition

$$-\,2\pi p = \frac{1}{J}\sum_{l=1}^{n_1}\frac{1}{q_{1,l}}\,. \tag{2.13}$$

Here n_k, m_k and p are all integers and reflect the ambiguity in the choice of branch for the logarithm. Finally, after the rescaling the expression for γ_1 takes the form

$$\gamma_1 = \frac{1}{8\pi^2 J}\sum_{j=1}^{n_1}\frac{1}{(q_{1,j})^2}\,. \tag{2.14}$$

3. Rational three-spin strings

In the following we shall consider the case of rational three spin strings. These can be reached by choosing

$$n_k = n\,, \qquad \forall \quad k \in \{1,\ldots,n_1\}\,, \tag{3.1}$$

$$m_k = m\,, \qquad \forall \quad k \in \{1,\ldots,n_2\}\,. \tag{3.2}$$

Then our Bethe equations trivially reduce to

$$\frac{1}{q_{1,k}} + 2\pi n = \frac{2}{J}\sum_{l\neq k}^{n_1}\frac{1}{q_{1,k}-q_{1,l}} - \frac{1}{J}\sum_{l=1}^{n_2}\frac{1}{q_{1,k}-q_{2,l}}\,, \tag{3.3}$$

$$2\pi m = \frac{2}{J}\sum_{l\neq k}^{n_2}\frac{1}{q_{2,k}-q_{2,l}} - \frac{1}{J}\sum_{l=1}^{n_1}\frac{1}{q_{2,k}-q_{1,l}}\,. \tag{3.4}$$

Similar equations appear in the study of the matrix model of $\mathcal{N}=2$ supersymmetric type $\mathrm{U}(N)\times\mathrm{U}(N)$ gauge theory [36]. In order to expose the similarity of the following analysis with the loop equation approach to matrix models we shall introduce the notation

$$V_1'(q) \equiv \frac{1}{q} + 2\pi n\,, \qquad V_2'(q) \equiv 2\pi m\,. \tag{3.5}$$

In addition, let us introduce the resolvents by

$$G_1(q) = \frac{1}{J}\sum_{j=1}^{n_1}\frac{1}{q-q_{1,j}}\,, \qquad G_2(q) = \frac{1}{J}\sum_{j=1}^{n_2}\frac{1}{q-q_{2,j}}\,, \tag{3.6}$$

and the filling fractions

$$\alpha = \frac{n_1}{J}\,, \qquad \beta = \frac{n_2}{J}\,. \tag{3.7}$$

Then we have

$$G_1(q) \sim \frac{\alpha}{q} \qquad \text{as } q \to \infty\,, \tag{3.8}$$

$$G_2(q) \sim \frac{\beta}{q} \qquad \text{as } q \to \infty\,, \tag{3.9}$$

as well as

$$G_1(0) = 2\pi p\,, \tag{3.10}$$

and

$$\gamma_1 = -\frac{1}{8\pi^2} G_1'(0)\,. \tag{3.11}$$

To determine γ_1 we first multiply eq. (3.3) by $\frac{1}{J^2}\sum_{m=1}^{n_2}\frac{1}{q_{2,m}-q_{1,k}}$ and sum over k. Next, we multiply eq. (3.4) by $\frac{1}{J^2}\sum_{m=1}^{n_1}\frac{1}{q_{1,m}-q_{2,k}}$ and sum over k and finally we subtract the resulting two equations. This gives us the following relation

$$-G_1'(0)\left(1-\frac{1}{J}\right) - 4\pi n G_1(0) + (2\pi n)^2\alpha + (G_1(0))^2 - (2\pi m)^2\beta = 0\,. \tag{3.12}$$

Furthermore, simply summing over k in eqs. (3.3) and (3.4) leads to

$$p = n\alpha + m\beta\,. \tag{3.13}$$

Bearing in mind the relations (3.10) and (3.11) and writing

$$\gamma_1 = \gamma_1^{(0)} + \frac{1}{J}\gamma_1^{(1)}\,, \tag{3.14}$$

we see that eq. (3.12) allows us to compute the conformal dimension as well as its leading non-anomalous $1/J$ correction. The leading order result for the conformal dimension reads

$$\gamma_1^{(0)} = \frac{1}{2}\left(n^2\alpha(1-\alpha) + 2\beta mn(1-\alpha) + m^2\beta(1-\beta)\right). \tag{3.15}$$

Translating from filling fractions (α,β) to angular momenta (J_1,J_2,J_3), cf. eq. (2.5) one sees that eq. (3.15) is exactly the expression characteristic of the general rational three-spin string upon the identification

$$m_1 = p\,, \qquad m_2 = p-n\,, \qquad m_3 = p-m-n\,. \tag{3.16}$$

The formula (3.15) and the accompanying constraint (3.13) are invariant under the transformations

$$1-\alpha \leftrightarrow \alpha-\beta\,, \qquad n\to -n\,, \qquad m\to m+n\,, \qquad p\to p-n\,, \tag{3.17}$$

$$\alpha \leftrightarrow 1-\beta\,, \qquad m \leftrightarrow -n\,, \qquad p\to p-m-n\,, \tag{3.18}$$

$$\alpha-\beta \leftrightarrow \beta\,, \qquad m\to -m\,, \qquad m\to m+n\,, \qquad p\to p\,. \tag{3.19}$$

These invariances are the generalizations of the invariance of the spin-1/2 Heisenberg chain under the interchange of spin up and spin down. The latter is recovered from relation (3.17) when $\beta = 0$, $m = 0$.

Including the $1/J$ correction simply gives $\gamma_1^{(1)} = \gamma_1^{(0)}$ or

$$\gamma_1 = \gamma_1^{(0)} \left(1 + \frac{1}{J}\right), \tag{3.20}$$

with $\gamma_1^{(0)}$ given above.

So far the Bethe root description of operators dual to rational three spin strings was given only for the case of two coinciding spin labels [10]. Such operators are of course included in the present analysis. Strings with angular momenta (J_1, J_2, J_2) where $J_1 > J_2$ and winding number k we can reach by choosing

$$m = -2n\,, \qquad p = 0\,, \qquad \text{and} \qquad m = k\,. \tag{3.21}$$

The momentum constraint (3.13) here implies

$$\alpha = 2\beta\,, \tag{3.22}$$

and therefore $(J_1, J_2, J_2) = J(1-\alpha, \frac{\alpha}{2}, \frac{\alpha}{2})$ where it is understood that $\alpha \in [0, \frac{2}{3}]$. Strings with angular momenta (J_1, J_1, J_3) where $J_1 > J_3$ and winding number k are reached by setting

$$n = -2m\,, \qquad p = -m\,, \qquad \text{and} \qquad n = k\,. \tag{3.23}$$

This choice of parameters implies

$$2\alpha = \beta + 1\,, \tag{3.24}$$

and thus $(J_1, J_1, J_3) = J(1-\alpha, 1-\alpha, 2\alpha - 1)$ where it is understood that $\alpha \in [\frac{1}{2}, \frac{2}{3}]$. In both of these cases we have

$$\gamma_1^{(0)} = \gamma_1^{(1)} = k^2 \frac{J_2}{J}\,. \tag{3.25}$$

We notice that the winding number of the string in the former case has to be identified with the mode number n while in the latter case it has to be identified with the mode number m, cf. eqs. (3.3) and (3.4). Strictly speaking, the expression (3.12) only constitutes a necessary condition for the Bethe equations to be fulfilled. However, we shall explain in section 5 how we can determine all the higher charges of the spin chain as well, thus obtaining the full solution of the Bethe equations (including the relation (3.12)).

Finally, let us mention that by setting

$$m = 0\,, \qquad \beta = 0\,, \tag{3.26}$$

and thus $p = n\alpha$, cf. eq. (3.13), we recover the result for the conformal dimension and the associated non-anomalous part of the finite size correction obtained in [30] for the case of a rational two-spin string with angular momentum assignment $(J_1, J_2, 0) = J(1-\alpha, \alpha, 0)$ and winding number n.

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4. Comparison to string theory

As already mentioned one sub-class of rational three-spin strings is particularly manageable, namely the sub-class of strings having two out of the three angular momenta coinciding, i.e. $(J_1, J_2, J_3) = (J_1, J_2, J_2)$. For such strings the lagrangian for the quadratic fluctuations around the classical solution involves only constant coefficients. This means that a stability analysis can immediately be carried out and in case of stability the one-loop correction to the energy can found as the sum of the characteristic bosonic and fermionic frequencies. The stability analysis gives a clear answer: the string is stable provided a parameter $q \approx \frac{2J_2}{J}$ fulfils the relation [3, 4, 29]

$$q < q_c = 1 - (1 - \frac{1}{2k})^2 , \tag{4.1}$$

where k is the winding number of the string. Translating to angular momenta this (approximately) reads

$$J_2 < \frac{1 - (1 - \frac{1}{2k})^2}{2(1 - \frac{1}{2k})^2} J_1 . \tag{4.2}$$

In particular, we see that the two-spin version of this string corresponding to $J_1 = 0$ is always unstable. For $J_1 \neq 0$ there is always a certain stability region and it appears that this stability region at least for $k \geq 2$ lies entirely within the class of strings that we can reach with the parameter choice given in eq. (3.21). (As it will appear from section 6 we also have access to the $k = 1$ case with the same parameter choice.) The actual computation of the one-loop string correction, i.e. the summing over bosonic and fermionic frequencies, is rather involved and can only be carried out numerically. We note that the bosonic frequencies can be reproduced in the gauge theory language by studying Bethe root fluctuations [8, 37]. It is, however, hard to see how the fermionic frequencies would be encoded in the Bethe root picture. The zero-mode contribution to the one-loop string correction is found to be [3, 4, 29]

$$E^{(1)}_{1,zero} = k^2 \frac{J_2}{J} . \tag{4.3}$$

This is identical to our result for the non-anomalous part of the $1/J$ correction $\gamma_1^{(1)}$, cf. eq. (3.25).

5. Higher charges

Above we have derived the one-loop conformal dimension including the non-anomalous finite size corrections for all rational three-spin strings by essentially a one line computation. In a similarly simple fashion one can derive expressions for the higher charges of the spin chain and the associated non-anomalous finite size corrections. The most efficient way to do so, however, is to determine in one step the resolvent $G_1(q)$ as it is well-known that this function acts as a generator of the higher charges [38, 10], see also [39]. It is straightforward to derive algebraic equations which determine $G_1(q)$ as well as $G_2(q)$. More precisely, we can derive a quadratic and a cubic equation which when combined determine the two

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resolvents. To derive the quadratic equation we first multiply eq. (3.3) by $\frac{1}{J}\frac{1}{q-q_{1,k}}$ and sum over k, subsequently multiply eq. (3.4) by $\frac{1}{J}\frac{1}{q-q_{2,k}}$ and sum over k and finally add the two resulting equations. This gives

$$G_1^2(q)+G_2^2(q)-G_1(q)G_2(q)--V_1'(q)G_1(q)-V_2'(q)G_2(q)+\frac{1}{q}G_1(0)+\frac{1}{J}\left(G_1'(q)+G_2'(q)\right)=0\,. \tag{5.1}$$

To derive the cubic equation we take a similar strategy. First, we multiply the relation (3.3) by $\frac{1}{J^2}\sum_{m=1}^{n_2}\frac{1}{q_{2,m}-q_{1,k}}\frac{1}{q-q_{1,k}}$ and sum over k, subsequently multiply the relation (3.4) by $\frac{1}{J^2}\sum_{m=1}^{n_1}\frac{1}{q_{1,m}-q_{2,k}}\frac{1}{q-q_{2,k}}$ and sum over k and finally we subtract the two resulting equations. The result of these manipulations reads

$$\begin{aligned}
G_2^2(q)G_1(q)-G_1^2(q)G_2(q)+V_1'(q)\left(G_1^2(q)+\frac{1}{J}G_1'(q)\right)-\\
-V_2'(q)\left(G_2^2(q)+\frac{1}{J}G_2'(q)\right)-(V_1'(q))^2G_1(q)+(V_2'(q))^2G_2(q)+\\
+\frac{1}{J}G_2'(q)G_1(q)-\frac{2}{J}G_1(q)G_1'(q)-\frac{1}{J^2}G_1''(q)-\frac{1}{Jq^2}G_1(q)+\\
+\frac{1}{q^2}G_1(0)+\frac{4\pi n}{q}G_1(0)+\frac{1}{q}G_1'(0)-\frac{1}{q}G_1^2(0)-\frac{1}{Jq}G_1'(0)+\frac{1}{Jq^2}G_1(0)=0\,.
\end{aligned} \tag{5.2}$$

A similar set of equations has been derived for the previously mentioned matrix model of $\mathcal{N}=1$ supersymmetric A_2 type $\mathrm{U}(N)\times\mathrm{U}(N)$ gauge theory [36]. These equations can easily be solved to leading and next to leading order in $1/J$. In particular, we can immediately discard the term involving $G_1''(q)$. Next, we have that all terms involving $G_1'(q)$ and and $G_2'(q)$ only appear to next to leading order in $1/J$. Thus in the large-J limit the equations (5.1) and (5.2) become equations involving only the resolvents themselves. Here one should remember that $G_1(0)$ as well as $G_1'(0)$ are known quantities, cf. eqs. (3.10) and (3.12). The quantity $G_1(0)$ is determined by the zero momentum condition, cf. eq. (2.13) and $G_1'(0)$ is determined by the asymptotic behaviour of the resolvents as $q\to\infty$. Actually, the relation (3.12) is nothing but the leading part of eq. (5.2) as $q\to\infty$. We note that the equation (5.1) in the large-J limit has a striking similarity with the loop equation of the $O(-1)$ matrix model and accordingly one can conveniently split the resolvents into their regular and singular parts as follows [40]

$$G_1(q)=\frac{1}{3}\left(2V_1'(q)+V_2'(q)\right)+g_1(q)\,, \tag{5.3}$$

$$G_2(q)=\frac{1}{3}\left(2V_2'(q)+V_1'(q)\right)+g_2(q)\,, \tag{5.4}$$

where $g_1(q)$ and $g_2(q)$ constitute the singular parts. Inserting this in eqs. (5.1) and (5.2) we find to leading order in $1/J$.

$$g_1^2(q)+g_2^2(q)-g_1(q)g_2(q)=r(q)\,, \tag{5.5}$$

$$g_1^2(q)g_2(q)-g_2^2(q)g_1(q)=s(q)\,, \tag{5.6}$$

where $r(q)$ and $s(q)$ are regular functions which read

$$r(q) = \frac{1}{3}\left((V_1'(q))^2 + (V_2'(q))^2 + V_1'(q)V_2'(q)\right) - \frac{G_1(0)}{q}\,, \tag{5.7}$$

$$\begin{aligned} s(q) = &\frac{2}{3}(V_1'(q) - V_2'(q))r(q) + \frac{8}{27}((V_2'(q))^3 - (V_1'(q))^3) + \frac{1}{9}V_1'(q)V_2'(q)\left(V_2'(q) - V_1'(q)\right) + \\ &+ \left(\frac{1}{q^2} + \frac{4\pi n}{q}\right)G_1(0) + \frac{1}{q}G_1'(0) - \frac{1}{q}G_1^2(0)\,. \end{aligned} \tag{5.8}$$

or more explicitly

$$r(q) = \frac{1}{3q^2} - \frac{2\pi}{q}\left\{n\left(\alpha - \frac{2}{3}\right) + m\left(\beta - \frac{1}{3}\right)\right\} + \frac{(2\pi)^2}{3}\left(m^2 + n^2 + mn\right), \tag{5.9}$$

$$\begin{aligned} s(q) = &-\frac{2}{27q^3} + \frac{2\pi}{3q^2}\left\{n\left(\alpha - \frac{2}{3}\right) + m\left(\beta - \frac{1}{3}\right)\right\} + \\ &+\frac{(2\pi)^2}{3q}\left\{n^2\left(\alpha - \frac{2}{3}\right) - m^2\left(\beta - \frac{1}{3}\right) + 2mn\left(\alpha - \beta - \frac{1}{3}\right)\right\} + \\ &+\left(\frac{2\pi}{3}\right)^3(m-n)\left(2n^2 + 2m^2 + 5mn\right). \end{aligned} \tag{5.10}$$

The singular parts of the resolvents thus fulfill the following cubic equations

$$(g_1(q))^3 - r(q)g_1(q) = s(q)\,, \tag{5.11}$$

$$(g_2(q))^3 - r(q)g_2(q) = -s(q)\,, \tag{5.12}$$

which can readily be solved. We notice that the functions $r(q)$ and $s(q)$ are again invariant under the transformations (3.17), (3.18) and (3.19). Thus all higher charges possess these invariances as well.

Let us write down the explicit expressions for $r(q)$ and $s(q)$ for the case where two out of the three spin labels coincide. For the choice of parameters given in eq. (3.21) and (3.22) i.e. for spin assignment $(J_1, J_2, J_2) = J(1-\alpha, \frac{\alpha}{2}, \frac{\alpha}{2})$ with $\alpha \in [0, \frac{2}{3}]$ and thus $J_1 > J_2$ we get

$$r(q) = (2\pi k)^2 + \frac{1}{3q^2}\,, \tag{5.13}$$

$$s(q) = -\frac{2}{27}\frac{1}{q^3} - (2\pi k)^2\left(\alpha - \frac{2}{3}\right)\frac{1}{q}\,. \tag{5.14}$$

For the choice of parameters given in (3.23) and (3.24), i.e. for spin assignment

$$(J_1, J_1, J_3) = J(1-\alpha, 1-\alpha, 2\alpha - 1) \qquad \text{with } \alpha \in \left[\frac{1}{2}, \frac{2}{3}\right]$$

and thus $J_1 > J_3$ we find

$$r(q) = (2\pi k)^2 + \frac{1}{3q^2}, \tag{5.15}$$

$$s(q) = -\frac{2}{27}\frac{1}{q^3} - (2\pi k)^2\left(2(1-\alpha) - \frac{2}{3}\right)\frac{1}{q}\,. \tag{5.16}$$

We notice that when expressed in terms of the doubly degenerate spin label, i.e. respectively $\alpha/2$ and $(1-\alpha)$ the two sets of equations (5.13), (5.14) and (5.15), (5.16) coincide. It is easy to see that the same is then the case for all the higher charges of the two string duals. A consequence of this is also that the expressions (5.13) and (5.14) remain valid if analytically continued to the formally forbidden parameter region $\alpha \in [\frac{2}{3}, 1]$ (and similarly for (5.15) and (5.16)).

The operators dual to rational three-spin strings with two out of the three spin labels coinciding were earlier studied in reference [10]. Here the starting point was for both situations a seemingly more specialized assumption about the root configuration. For spin assignment (J_1, J_1, J_2), $J_1 > J_2$ the roots $\{q_{1,j}\}_{j=1}^{n_1}$ were assumed to be living on two distinct contours, for which the associated mode numbers were respectively k and $-k$, and which were each others mirror images with respect to the imaginary axis. Furthermore, the roots $\{q_{2,j}\}_{j=1}^{n_2}$ were supposed to spread out over the entire imaginary axis. This meant that the rational three-spin string in question was effectively reached as a limiting case of an elliptic three-spin string [11]. From the double contour assumption an equation for the leading J contribution to the singular part of the resolvent for the roots $\{q_{1,j}\}_{j=1}^{n_1}$ was derived. This equation is *exactly* identical to (5.11), (5.13) and (5.14). Thus, all charges of the Bethe state of [10] agree with those of ours to leading order in $1/J$. The two states are therefore indistinguishable as descriptions of the classical three-spin string. The case (J_1, J_1, J_3) with $J_1 > J_3$ was treated in reference [10] starting from a root configuration which included a so-called condensate. Also in this case the rational three-spin string in question was effectively obtained as a limiting case of an elliptic three-spin string [12]. The condensate assumption did not immediately lead to a simple equation for the resolvent. It was nevertheless suggested that the condensate solution was the analytical continuation of the double contour solution to the formally forbidden region of parameter space. Here, we have obtained a simple equation for the resolvent and we have seen explicitly how the analytical continuation works. While it is well-understood in terms of algebraic geometry that the assumption of a condensate is redundant [14], it is less clear why the double contour assumption of [10] for the case (J_1, J_2, J_2) with $J_1 > J_2$ is equivalent to the treatment given here.

We note that having obtained the leading order contributions to the resolvents it is straightforward to determine the non-anomalous part of the next to leading order ones as well. This simply requires expanding the singular parts of the resolvents as follows

$$g_1(q) = g_1^{(0)}(q) + \frac{1}{J}\, g_1^{(1)}(q), \qquad g_2(q) = g_2^{(0)}(q) + \frac{1}{J}\, g_2^{(1)}(q). \tag{5.17}$$

6. Particular points

Above we have derived the conformal dimension including the non-anomalous part of the finite size correction for all rational three-spin strings by a one line computation. One word of caution is needed, though. Our expression for γ constitutes a necessary condition for the Bethe equations to be fulfilled. Implicitly we have assumed that no root of the first type coincides with a root of the second type. We need to check that the full solution of the Bethe equations indeed has this property. In the thermodynamic limit, $J \to \infty$, the Bethe roots condense on smooth contours in the complex plane. Thus we should investigate whether the contours corresponding to $\{q_{1,j}\}_{j=1}^{n_1}$ and $\{q_{2,j}\}_{j=1}^{n_2}$ are indeed disjunct. The supports for the distributions of roots constitute the branch cuts of the resolvents $G_1(q)$ and $G_2(q)$. To determine the branch points of the resolvents we only need to consider the $J \to \infty$ limit of eqs. (5.1) and (5.2). The branch points of the resolvents are thus given by the single zeroes of the common discriminant of the two cubic equations (5.11) and (5.12). The discriminant reads

$$\Delta = 4(r(q))^3 - 27(s(q))^2 \,. \tag{6.1}$$

Inserting eqs. (5.9) and (5.10) in eq. (6.1) we find that the leading and next to leading order term in $1/q$ cancel out so that Δ becomes a polynomial of degree four in $1/q$. The four zeroes of this polynomial then constitute the branch points for $G_1(q)$ and $G_2(q)$. In general one can not determine the precise location of the cuts from the position of the branch points alone. In order that a meaningful density of Bethe roots can be associated with the resolvents the discontinuity of these across their respective cuts, i.e. $\rho(q) = \frac{1}{2\pi i}\{G(q+i0) - G(q-i0)\}$, must fulfill that $\rho(q)dq$ is real and positive [41]. There are in general three directions consistent with positivity from which a cut can emerge from a branch point [41]. We expect that in general it will be possible to choose the cuts so that they do not overlap. (For an example of how this works, see [42]). There is, however, one situation where we can detect a signal of our solution coming to a limitation, namely the situation where two or more of the four branch points coincide. This either means that one (or two) cuts degenerate to a point or that two cuts touch each other. As the general expression for Δ is rather involved, let us specialize to the cases where two spin labels coincide.

Let us consider the case $\beta = \alpha/2$, $n = -2m = k$ corresponding to a string with spin assignment (J_1, J_2, J_2), $J_1 > J_2$ and winding number k. This is the case which includes the stable three-spin string. Here, the allowed region for α is $0 \le \alpha \le \frac{2}{3}$. In this case we get for the discriminant

$$\Delta \propto (1-\alpha) - q^2(8 - 36\alpha + 27\alpha^2)(k\pi)^2 + q^4(2\pi k)^4 \,. \tag{6.2}$$

This polynomial has two sets of double roots at $q = \pm i\sqrt{3}(2\pi k)$ when

$$\alpha = \frac{8}{9} \,. \tag{6.3}$$

This value of α was likewise singled out in the analysis of [10] for reasons similar to the one described here. We notice that the particular point $\alpha = 8/9$ lies outside the allowed region (and thus does not affect our comparison with string theory). Exploiting the symmetry of the equation (5.14) and (5.16) under $\alpha/2 \to 1-\alpha$ we immediately find that for a string with spin assignment (J_1, J_1, J_3) with $J_1 > J_3$ the corresponding particular value of α is

$$\alpha = \frac{5}{9} \,. \tag{6.4}$$

This value of α was likewise singled out in the analysis of [10]. We expect to have an entire line of particular points $\alpha = \alpha_*(\beta)$ in the parameter space.

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7. Conclusion

We have determined the one-loop conformal dimension and the associated non-anomalous finite size corrections for all operators dual to rational three-spin strings. The conformal dimensions match the classical string energies and for operators dual to stable three-spin strings with two coinciding angular momenta the non-anomalous part of the finite size correction agrees with the zero-mode contribution to the one-loop string sigma model energy. Very recently there was another comparison of gauge theory non-anomalous finite size corrections and zero-mode contributions to string one-loop energies which likewise resulted in agreement [31]. This comparison concerned a stable two-spin string with one large angular momentum on S^5 and one on AdS_5. We expect that it should be relatively straightforward to extend our analysis of the three-spin case to the situation where one or two of the spins lie in AdS_5 in stead of S^5. It appears less straightforward to treat the case of elliptic or hyper-elliptic strings. Rather than extending the analysis to further particular solutions it would of course be more interesting to find a unifying geometric description of the finite size corrections applicable to any string solution — in the spirit of the treatment of the leading order contribution [14, 19 − 21].

As mentioned in the introduction, it has been known for some time that to leading order in $1/J$ there exists a discrepancy between the classical energy of spinning strings and the conformal dimension of the dual operators at third order in λ/J^2 [25], see also [26]. For near BMN states a similar three-loop disagreement is observed between gauge- and string theory at next to leading order in $1/J$ [22, 43]. Therefore, for spinning strings at next to leading order in $1/J$ one would likewise expect a disagreement to turn up at some low order in λ/J^2. Our results as well as the results of [31] indicate that this does not happen at first order in λ/J^2 and the forthcoming paper [32], which takes into account the anomalous part of the finite size correction, find complete agreement at this order for the case of a two-spin string with one large angular momentum on S^5 and one on AdS_5. It would of course be interesting to determine the anomalous part of the finite size correction for the three-spin strings as well. This would require the explicit knowledge of the leading order Bethe root distributions, an information which is in principle accessible but in practice hard to obtain, cf. section 6. In any case, we have with our investigations enlarged the region of parameter space where a comparison of semi-classical strings and their dual operators is possible and have as well provided new data which might help in further refining the methods for comparing gauge- and string theory.

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